TIME-DEPENDENT SCATTERING BY A SOUND-HARD SPHERE

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Motivation

Time-dependent scattering in three dimensions is a difficult mathematical problem to solve analytically and numerically. Analytical solutions obtained using the method of separation of variables is complicated, and numerical methods to solve this problem have challenges for computing solutions at long distances and long times. We seek a simple, effective, and efficient method to solve this problem based on the method of fundamental solutions. The specific case of a sound-hard sphere is considered in the context of acoustics.

Background

- Consider a sphere of radius a centered at the point r_c. We seek a method to solve the scattering problem for the three-dimensional wave equation corresponding to this sphere:

$$u_{tt} + c^2 \Delta u = 0 \text{ in } |r - r_c| > a \text{ and } t > 0$$
$$u(r, 0) = f(r),$$
$$u_t(r, 0) = g(r)$$
$$\partial_n u = 0 \text{ on } |r - r_c| = a \text{ and } t > 0$$

  where c denotes the wave speed and $\partial_n \equiv \nabla u \cdot \hat{n}$ denotes the normal derivative.
- The condition on the normal derivative($\partial_n u = 0$) gives the sound-hard condition.
- The solution of this scattering can be expressed as the sum of an incident field and scattered field

$$u = u^{inc} + u^s.$$

- The sound-hard condition can be reformulated in terms of these fields:

$$\partial_n u^s = -\partial_n u^{inc}.$$

- For the purposes of this analysis, the incident field is taken to be $u^{inc} = \cos(z - ct)$, which describes a set of plane waves propagating along the z-axis. We use the sound-hard boundary condition, among other things, to determine the scattered field u^s.

Methodology

- We use the method of fundamental solutions[2] whereby we approximate the scattered field by a superposition of finitely many spherical waves with unknown coefficients:

$$u^s(r,t) \approx \sum_{j=1}^{NT} \sum_{i=1}^{P} c_{ij} v_{ij}(r - r_i^c, t),$$

$$v_{ij}(r - r_i^c, t) = \begin{cases} \dfrac{1}{4\pi c^2 (r - r_i^c)}, & if\ t\ \in [t_j + \dfrac{r - r_i^c}{c}, t_{j+1} + \dfrac{r - r_i^c}{c}] \\ 0, & otherwise \end{cases}$$

- Each spherical wave v_{ij} radiates from a point source centered at r_i^c within the sphere.
- In order to fully solve the scattering problem, we must
  1. construct the sphere of radius a with the optimal placement of these point sources, and
  2. determine the time-dependent coefficients $c_{ij}(t)$ as to satisfy the sound-hard boundary condition.
- This optimal placement of the sources is achieved by mapping the sources unto the sphere according to a Fibonacci lattice structure[1]; the sources are then re-centered within the sphere in accordance with the domain of the initial-value problem(see Fig.1).
- The coefficients are determined by solving the linear system of equations that arises from evaluating the boundary condition at each source(1 to P) for each discrete time interval(1 to NT).
- Once the coefficients are determined, the scattered field can be plotted with the use of MATLAB(see Fig. 2). In order to verify this approximation, we compare it to the time-harmonic analytic solution that is found by solving the Helmholtz equation $\Delta U + k^2 U = 0$ in spherical coordinates(see Fig. 3).

Results

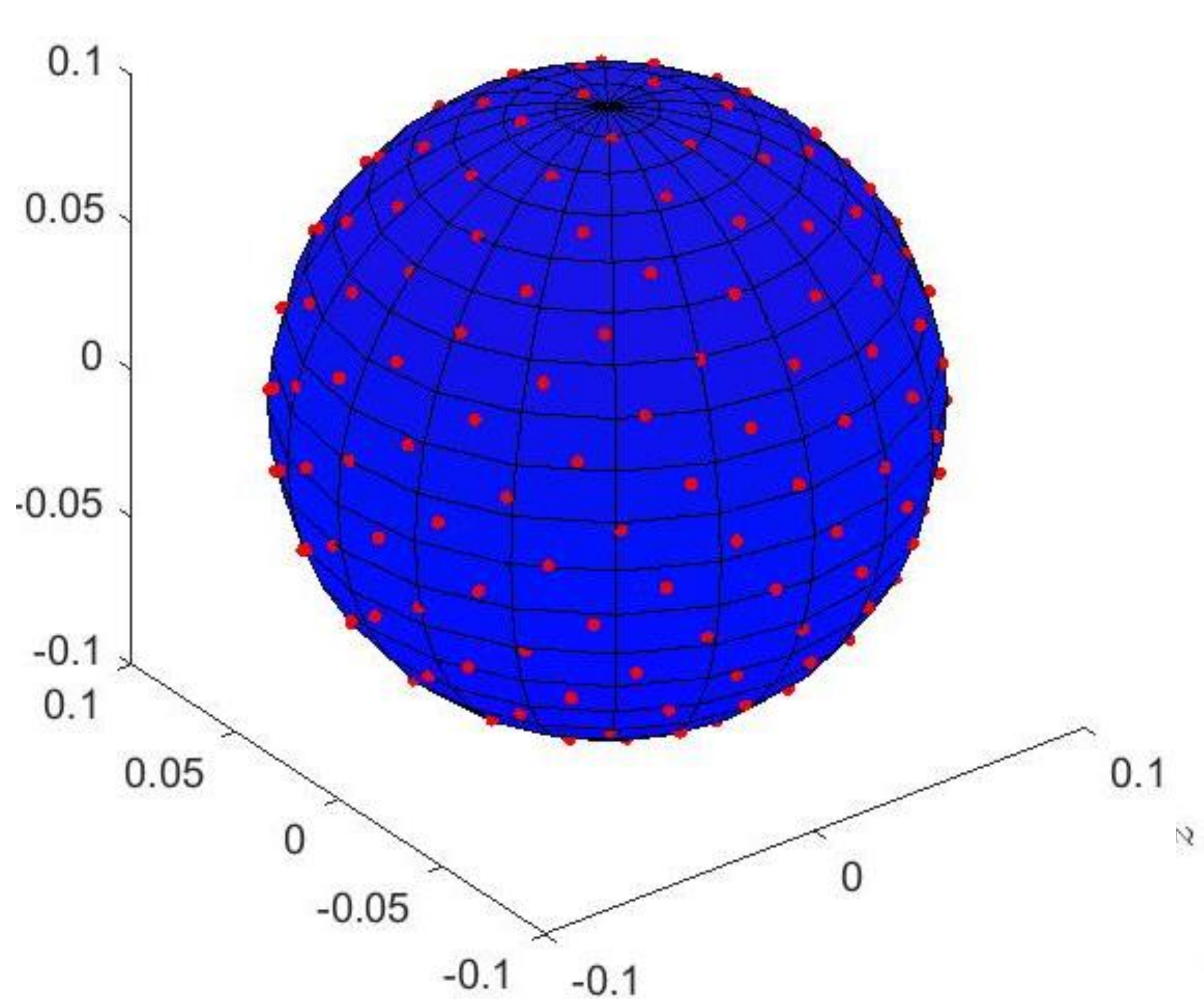

Figure 1: Scattering sphere with sources(red) arranged in a Fibonacci lattice. $a = 0.1, P = 209.$

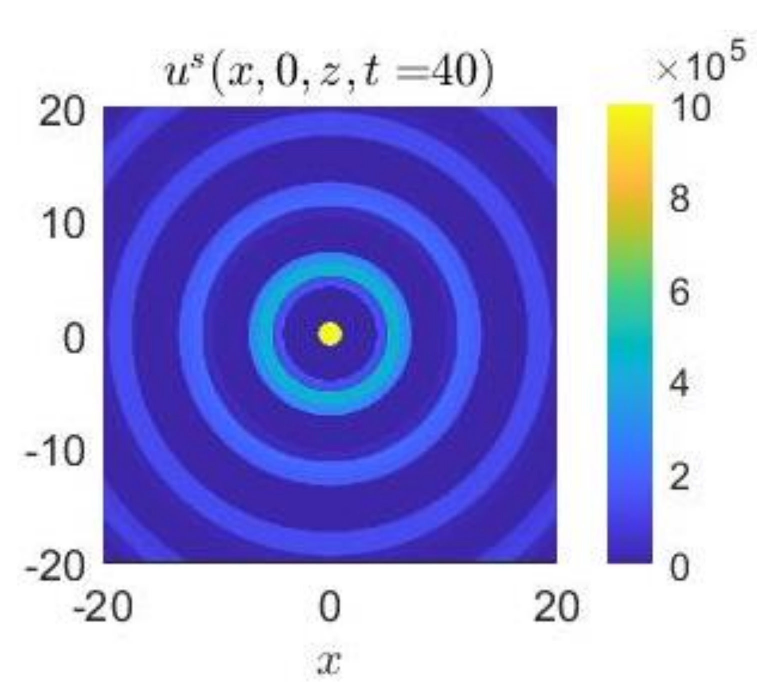

Figure 2: xz-plane colormap of scattered field u^s. $a = 0.1, P = 209, N = 50$

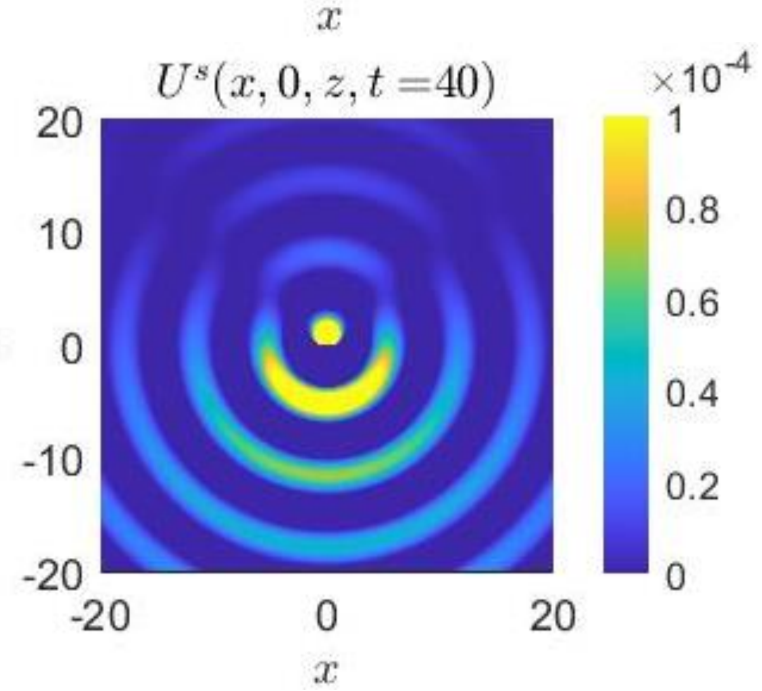

Figure 3: xz-plane colormap of time-harmonic analytic solution

Conclusions

- Overall, our results demonstrate the usefulness of the application of the method of fundamental solutions to scattering problems.
- By comparing Figures 2 and 3, our approximation of the scattered field is at least visually similar to the field derived from solving the Helmholtz equation analytically.
- Further work must be done in order to precisely quantify the error between the two fields. Moreover, the application of this approximation to different scattering problems must also be considered for future studies. It would be important to study how changes in the parameters, such as the number of sources and their arrangement, correspond to different scattering scenarios.

References

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