Solving PDEs with the Finite Element Method using Firedrake

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Abstract

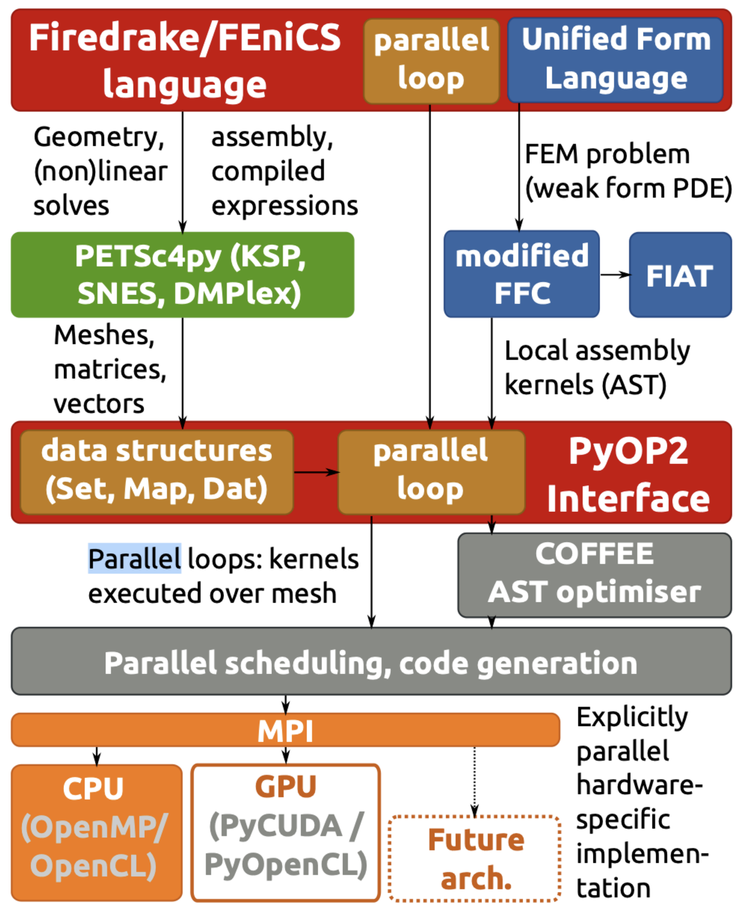
Partial Differential Equations (PDEs) are used to model a wide variety of phenomena such as sound, heat, diffusion, fluid dynamics, elasticity, etc. While some of these equations can be solved analytically, in most cases one needs to implement a numerical method such as the finite element method (FEM) to solve them numerically. Firedrake is a FEM-based automated system for solving PDEs numerically that can be used to alleviate implementation burden.

Goal of the project: The main goal of this project is to investigate Firedrake's potential as a PDE solver for inverse problems governed by PDEs. In particular we:

1. Investigated how to install and run Firedrake.
2. Implemented the Poisson model problem.
3. Studied the convergence for the numerical solution of the Poisson problem with linear and quadratic finite element basis functions.

Firedrake: PDE Toolbox

The Firedrake software stack is made up of various packages and layers, which are used together to compose models of finite element problems. The following illustration[3] shows the internal Firedrake toolchain.



Solving the Poisson problem with Firedrake

The Poisson equation is a prototype elliptic partial differential equation. Let Ω and $\Gamma = \Gamma_N \cup \Gamma_D$ ($\Gamma_N \cap \Gamma_D = \emptyset$) be the domain and boundary, respectively.

Poisson Problem:

$$\begin{aligned} -\nabla\cdot(\kappa\nabla u) &= f, && \text{in } \Omega \\ \nabla u\cdot\boldsymbol{n} &= g, && \text{on } \Gamma_N \\ u &= u_0, && \text{on } \Gamma_D \end{aligned}$$

Input Parameters:

$$f(x,y) = 2\pi^2[\sin^2(\pi x)\cos(2\pi y) - \cos(2\pi x)\cos^2(\pi y)]$$
$$g(x,y) = 0$$
$$u_0(0,y) = u_0(L,y) = 0, \kappa = 1, L = H = 1$$

True Solution:

$$u_{\text{true}}(x,y) = \sin^2(\pi x)\cos^2(\pi y)$$

Function Spaces:

$$L^2(\Omega) := \left\{\forall v\ :\ \int_\Omega |v(x)|^2 dx < \infty\right\}$$
$$\mathcal{H}^1(\Omega) := \{v \in L^2(\Omega)\ : \partial v/\partial x_i \in L^2(\Omega)\}$$
$$\mathcal{V} := \{v \in \mathcal{H}^1(\Omega)\ : v = u_0 \text{ on } \Gamma_D\}$$
$$\mathcal{V}_0 := \{v \in \mathcal{H}^1(\Omega)\ : v = 0 \text{ on } \Gamma_D\}$$

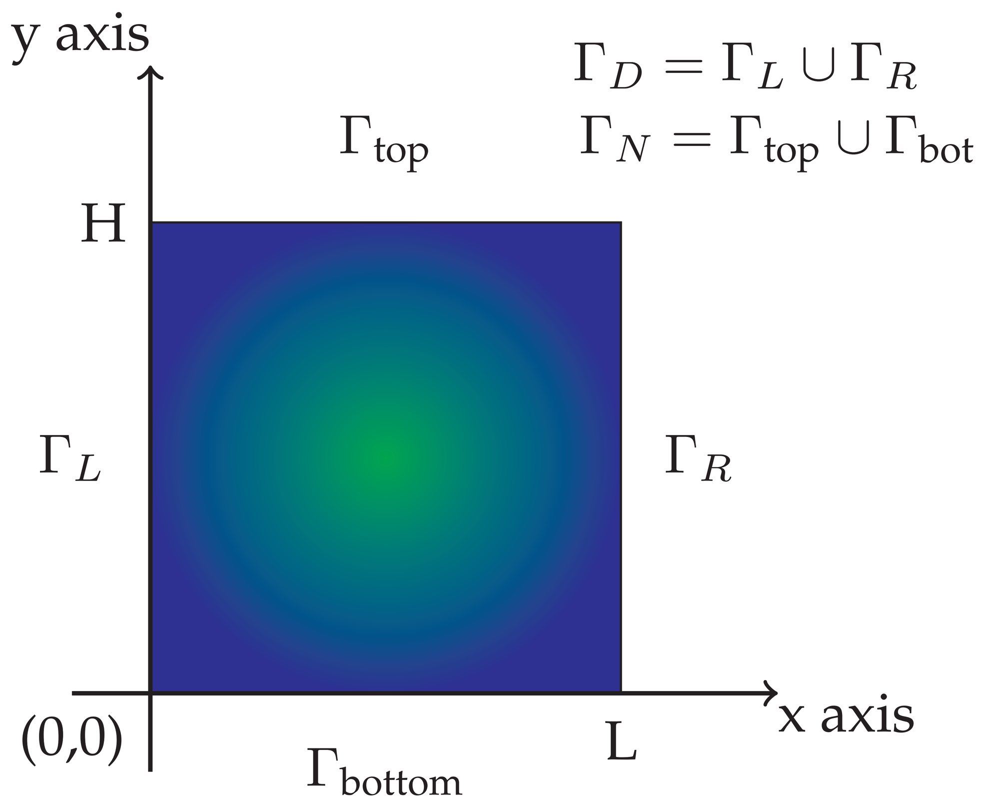


Figure 1: The domain and boundaries

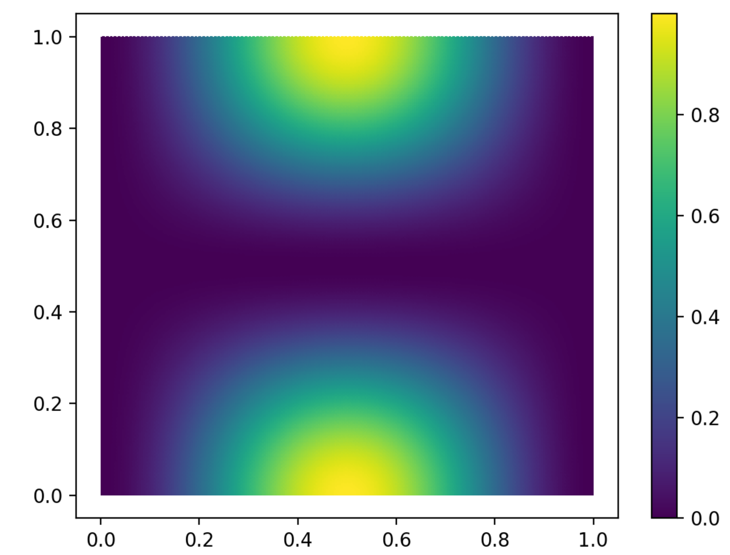


Figure 2: The true solution $u_{\text{true}}(x,y)$

The Poisson equation can be formulated in a variational form; that is, a form where the variable u can be approximated numerically while in the same time we relax some of the regularity conditions.

Variational Formulation of the Poisson Problem

Find $u \in \mathcal{V}$ such that

$$\int_\Omega \kappa\nabla u\cdot\nabla v\,\mathbf{d}\Omega = \int_\Omega fv\,\mathbf{d}\Omega + \int_{\Gamma_N} gv\,\mathbf{d}\Gamma_N$$

for all test functions $v \in \mathcal{V}_0$.

Numerical Solution

Convergence study				
N	L^2-Error	Rate	E-Error	Rate
16	6.52e-3 1.61e-4	3.76 7.87	2.79e-1 1.88e-2	1.95 3.87
32	1.66e-3 2.02e-5	3.94 7.94	1.40e-1 4.76e-3	1.99 3.96
64	4.16e-4 2.54e-6	3.98 7.97	7.03e-2 1.19e-3	2.00 3.99
128	1.04e-4 3.41e-7	4.00 7.45	3.52e-2 2.99e-4	2.00 3.99

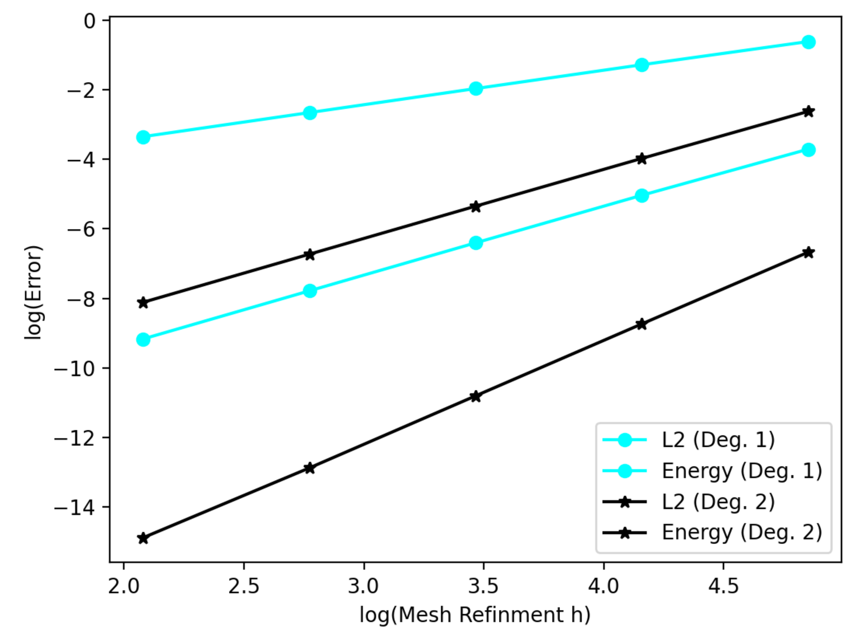


Figure 3: Log-log plot of the errors $\|e\|_E$, $\|e\|_{L^2}$ as functions of h for the Poisson problem for piecewise-linear (blue) and quadratic (black) basis functions.

Conclusions & Future Work

- Firedrake offers a user friendly and comprehensive installation process, with detailed instructions/documentation.
- Capabilities include: weak forms, derivatives, access to PETSc linear algebra solvers, visualization with Paraview, etc.
- The numerical results comply in accordance to the expected theoretical convergence rates.
- Future work includes extending the Poisson code to an inverse problem and to investigate the parallelism capabilities of Firedrake.

Accuracy of the FEM Approximation[2]

- u_h: numerical sol. for mesh size h
- $e = u - u_h$: FEM approximation error
- p: the power of the finite element basis functions; it measures *the rate of convergence*
- C_1, C_2, C_3: constants independent of h

A-priori error estimates [?]:

$$||e||_E = \frac{1}{2}\int_\Omega \kappa\left[\frac{de}{dx}\right]^2 dx \le C_1 h^p \qquad \text{Energy Norm}$$

$$||e||_{L^2} = \left(\int_\Omega e^2 dx\right)^{1/2} \le C_2 h^{p+1} \qquad L^2 \text{ Norm}$$

$$||e||_\infty = \max_{x\in\Omega} |e(x)| \le C_3 h^{p+1} \qquad \text{Uniform Norm}$$

References

[1] The Firedrake Team. "Firedrake Documentation.", https://www.firedrakeproject.org/documentation.html.

[2] Becker, E. B., Graham F. Carey, and J. T. Oden. Finite Elements : An Introduction. Volume 1. Prentice Hall, Englewood Cliffs, N.J., 1981.

[3] Florian Rathgeber, David A. Ham, Lawrence Mitchell, Michael Lange, Fabio Luporini, Andrew T. T. McRae, Gheorghe-Teodor Bercea, Graham R. Markall, and Paul H. J. Kelly, 2015. Firedrake: automating the finite element method by composing abstractions. ACM Trans. Math. Softw. 0, 0, Article 0 (0), 28 pages.